1. Introduction:

Symmetry is universal, fascinating, and of immense practical importance. As human beings, we have evolved a perception of symmetry that lies at the core of our conscious life. Symmetries provide cues that help us relate to our environment and guide our movements through the world. Everyone has a taste for things that are in some way symmetrical or possess a pleasing deviation from perfect symmetry.

Perhaps the most-perfect object we can imagine is a circle; yet dividing the circumference by the diameter produces the irrational number π that we can only symbolize. Perfect, unequivocal, symmetry, like perfect theory, eludes us always.

A symmetry group of a system of differential equation is a group of transformations, which maps any solution to another solution of system. According to Lie's (1975) frame work such a group depends on continuous parameters and consist of either point transformation (point symmetry) acting on a systems space of independent and dependent variables, or their generally, contact transformations (contact symmetry) acting on the space including all first order derivatives of dependent variables. Elementary examples of Lie groups include translation, rotation and scaling and autonomous system of first order ordinary differential equation (stationary flow) essentially defines one parameter Lie group of point transformation whereas discrete groups of point transformation, acting on the space of its dependent and independent variables that can be determine by Lie's algorithm.

Mathematicians have studied the theory of continuous transformation of group extensively since the late of nineteenth century. First, it was Sophus Lie (1842-1899), the Russian mathematician, who has introduced the notations continuous groups, known as Lie groups, in order to unify and extend various specialized solution methods for the ordinary differential equations. In fact, Lie was inspired by lectures of Sylow (Birkeland, 1996) given at Christiania (present day Oslo) on Galois Theory and Abel's related works in 1881. Sylow and Lie collaborated in a careful editing of Abel's complete work. One can reduce lie show then the order of ordinary differential equation, constructively, if the said differential equation is invariant under a one-parameter Lie group of point transformation.

Lie's work systematically relates miscellany of topics in ordinary differential equations including integrating factor, separable equation, homogeneous equation, reduction of order of an equation, method of undetermined coefficient and the method of variation of parameter for linear equation, Euler's equation and the use of Laplace

transform. Further Lie has also indicated that linear partial differential equation invariant under a group, so called Lie group, leads directly to superposition of solutions in terms of transformations.

Although research in this area is steel, being pursued practical application of the fundamental theory has received limited attention. Our motivation of the present investigation is to extend pertinent theories to the development of an elegant procedure for the solution of boundary value problems represented by non-linear differential equation governing the different physical situations arising in science and engineering.

In the Real world as well as of the Universe all the substances are classify by the four states like solid, liquid, gas and plasma. Among these, the later three states liquid, gas, and plasma are referred as 'fluid '. Alternately, in a scientific terminology, a fluid is a substance that deforms continuously under the application of a shear stress, no matter how small the shear stress may be. In the virtue viscosity, fluids are classified into two classes:

1. Real fluids, those who are having viscosity and
2. Ideal fluids, those who are supposed to have no viscosity, surface tension and compressibility

Almost all the matters in the universe are in fluid state. Interstellar and intergalactic space is filled with gases. The stars consist of gases and liquid. Even the earth enveloped by gaseous atmosphere has a liquid core and about two third part of its surface is covered by water.

The subject of Fluid Mechanics is the study of fluids, which deals with the behavior of fluids at rest and in motion. Now days, fluid mechanics plays an important role in the all area of education like Science and Engineering, Astrophysics, Biology, Metrology, Physics, Chemistry, Geophysics etc. Also it is easy to appreciate the principles of fluid mechanics when we observe a few natural phenomena like the flight of birds in air, the motion of fish in water and circulation of blood in human body.

Fluids are encountered at each step of life. While some fluids are Newtonian in behavior and others come under Non-Newtonian type. The fluid for which the relationship between shearing stress and the rate of deformation is linear is said to be Newtonian fluid. Moreover, a fluid that does not obey Newton's law of viscosity is said to be Non-Newtonian fluid. In Non-Newtonian fluids, shearing stress and rate of deformation are not linearly related. These types of fluids are found to be of great commercial importance. They are handled extensively by chemical industries namely plastics and polymers.

Biological and biomedical devices make use of the rheological aspect of non - Newtonian fluids. The wide usage and application of these fluids in the diverse industrial fields have prompted modern researchers to explore extensively the field of non - Newtonian fluids.

2. Literature Review:

We can find several types of Non-Newtonian fluid models that are proposed by scientists working in this area. Several empirical models are generally used to approximate the experimental data are available for such fluids. It may not be possible to list all such models exhaustively. However, some of them are listed below: (Bird et. al., 1960), (Wilkinsion, 1960)

Power-law fluids, Eyring fluids, Sisko fluids, Prandtl - Eyring fluids, Sutterby fluids, Prandtl fluids, Ellis fluids, Williamson fluids, Reiner- Philippoff, Powell-Eyring fluids etcetera.

A well-structured flow characterized by smooth and organized motion of fluid particles in laminar or well-defined layers, usually occurring at relatively small Reynolds number is Laminar. In addition, an unstable / agitated flow characterized by unorganized motion of fluid particles in random direction, usually occurring at higher Reynolds number is said to be turbulent flow. In other words, viscous flow regimes are classified as laminar or turbulent on the basics of flow structure. In the laminar regime, flow structure is characterized by smooth motion in laminar or layers. Flow structure in the turbulent regime is characterized by random motion.

Generally, fluids are classified as liquids or gases, which may be electrically conducting such as mercury or ionized gas. The electrical conductivity of the fluid plays, its role, only when, fluid is moving in an electromagnetic field, otherwise the fluid may be regarded as an ordinary fluid (non-electrically conducting fluid). Magneto fluid dynamics (MFD) is the study of the motion of an electrically conducting fluid in the presence of a magnetic field. When the fluid is incompressible, such as liquid mercury, and its other properties such as viscosity, thermal conductivity and electrical conductivity are regarded as constant the word magnetohydrodynamics (MHD) or hydro-magnetic is used. In two-dimensional boundary layer flows, almost exclusively considered up to the present time, all flow parameters are independent of the third coordinate.

The mathematical formulation of the most of physical phenomena leads to the differential equations. Most of these differential equations are non-linear partial

differential equations (PDEs) and in order to study a real physical situation one needs to solve these equations. It is often hard to solve these non-linear partial differential equations. Indeed, there is no full proof method (general method) available even to solve a class of such a PDE even if it looks quite simple.

Further it is to be observed from the survey of literature that most of the analyses regarding the fluid flow are analyzed for the Newtonian fluid where as the analysis for Non-Newtonian fluids are made by considering limited Non-Newtonian fluid, almost the Power-Law fluid. Although there are, various types of Non-Newtonian fluids are available, for the applications in real world. In the present thesis, we have attempted to analyze the different types of typical Non-Newtonian fluids by applying the similarity analysis via group theory method.

The method of searching particular type of transformations, which transforms the partial differential equation into ordinary differential equation, is known as similarity method. That is the term "Similarity transformation" or “Symmetry” of a partial differential equation shall be defined to be a transformation of independent and dependent variables occurring in the equation such that the number of independent variables appearing in the transformed equation is at least on less than in the original equation. If a partial differential equation has two independent variables, a similarity transformation would transform the equation into an ordinary differential equation. Helmholtz (1973) first discovered the concept of ‘Mathematical Similarity’ through the dimensional analysis approach. In the same time, the Russian mathematician S. Lie (1975) has presented famous Lie algorithm based on the continuous group transformations for the same kind of simplification of partial differential equation. Later on importance of similarity parameters in physical sciences was demonstrated by Reynolds (1895), Blasius (1908), Buckingham (1914), Sedov (1959) and developed the concept of similarity in a more meaningful way till Morgan (1952) following the concepts of Birkhoff (1950). That showed that the boundary layer equations were invariant under certain group of transformations, which reduced the number of independent variables by one. The power of various similarity techniques is recently demonstrated by various authors like Hansen (1964), Moran and Gaggioli (1968), Seshadri and Na (1985), Rogers and Ames (1989), Bluman and Kumai (1989), and Drachner (1999).

Today, there are number of symmetry (Similarity) techniques found in literature out of which following four similarity methods are frequently found in the literature. These symmetry methods are,

1. Free Parameter
2. Separation of Variables
3. Group Theoretic
    a. Deductive Method (Modern Group Method)
    b. Lie Group- Method (Classical Method)
4. Dimensional Analysis

Among these methods group theoretic is more attractive than the other methods because this method derived systematically from simple group transformations laws.

There are many extraordinary and splendid applications of group theory. If a system of partial differential equation is invariant under group of point transformation, one can find, constructively special solution, called similarity solution or invariant (unchanged) solution, which are invariant under same subgroup of the full group admitted by the system. Ovsiannikov (1962-1982) recognized this application of group theory in the late 1950's through the work of Soviet group of mathematicians lead. Invariant solution can also be constructed for specific boundary value problems. Bluman and Cole (1974) have derived method to introduced subgroup of the full group of a given partial differential equation that leaves boundary curves and condition in post on them invariant. Such solutions include self-similar (autonomous) solutions, which can be obtain through dimensional analysis more generally from invariants under groups of scaling.

The classical application of simple Lie group transformation, one parameter transformation of group and their use in obtaining invariant solution of differential equations found almost everywhere in science and engineering. The classical treats on such techniques are found in Bluman and Cole (1974), Cohen (1911). More recently, numbers of research papers on the applications of group transformation in fluid mechanics, bio-fluid mechanics, in civil engineering are found in literature.

The power of various similarity techniques are demonstrated by various authors like Hansen (1964), Bluman and Cole (1974), Seshadri and Na (1986), Bluman and Kumai (1985), and more recently by Bluman and Anco (2002). The various applications of different similarity techniques are given by Hansen and his Co-workers (1956, 1956a, 1967, 1967a), Pakdemirli and his co-workers (1992, 1993, 1994, 1997, 1999, 2002), Malek

and his co-workers (1990, 1991, 2004, 2006, 2008) and Timol and his co-workers (1986, 1987, 1988, 1998, 2004, 2008, 2009).

In this thesis, we shall discuss the modern deductive group-symmetry techniques exclusively for the area of fluid mechanics, especially for the non-linear partial differential equations, governing the flow of Non-Newtonian fluids.

3. Outline of the Thesis:

In present thesis, Similarity solutions of steady and unsteady, two-dimensional, incompressible, laminar boundary layer flows, MHD boundary layer flow for Non - Newtonian fluids of different models are discussed using deductive group theoretic method in detail. The entire research work is presented in seven well planned chapters. The successions of these chapters are embedded as follow:

Chapter 1:

In this chapter, we shall build up a strong structure in logical manner to provide knowledge of very basic concepts of fluid mechanics and magneto hydrodynamics. It is the essential part of study to have better understanding of behavior of particle in flow. This chapter deals with definitions and a brief discussion of Newtonian and Non-Newtonian fluids. In addition, contains some useful information about the rheological development of different Non--Newtonian fluid models. The basic governing equations for the fluid flow are also discussed in this chapter, like equation of continuity, mass conservation law, energy conservation law etc. Clearly, this chapter provides strong foundation for next coming chapters.

Chapter 2:

This chapter deals with the complete procedure so called deductive group theoretic technique. In this chapter, we have introduced the deductive group theoretic method, based on general group of transformation, to find the similarity solution of flow problems. This method is based on group theory. We have also explained how to form a continuous deductive group transformation for similarity analysis and to search a sub group of transformation that transformed invariantly the governing non-linear partial differential equations in to the ordinary differential equation, by means of basic theorems of group theory. The entire technique is also illustrated by considering a suitable example.

Chapter 3:

This chapter deals with the derivation of the fundamental equations for MHD flow of a Sisko fluid past through infinite porous rigid half-space. These equations are transformed to similarity equations using deductive method, (discussed in chapter 2). Further the transformed similarity solutions are obtained for various numerical values of parameters using MATLAB coding. Then the normalized velocity profiles are generated for the different flow parameters. Finally, the effects of various parameters on Newtonian and Non-Newtonian fluid flow are studied. The remarks on behavior of velocity profiles are quite useful in the practical problems involving Newtonian and Non-Newtonian fluids with different flow index in engineering and science.

Chapter 4:

In this chapter, we have derived the similarity solutions of laminar incompressible boundary layer equations of all Non-Newtonian viscoinelastic fluids that are characterized by the property that its shearing stress is related to the rate of strain by some arbitrary continuous function. It is interesting to note that we are first to apply the modern deductive group theoretic technique based on general group of transformation to derive proper similarity transformations for the non-linear partial differential equation governing all those Non-Newtonian fluids which are characterized by the property that its stress-strain are related by some general arbitrary continuous functional relationship. Further, it is to be observed that similarity solutions for all such Non-Newtonian fluids exist only for the flow past 90° wedge only. The present similarity equation is solved by shooting method, using MATLAB coding for the case of Prandtl-Eyring fluid model. From the numerical solution it can be interpret ate that the flow parameters, which are exist in constitutive equation, play an important role in fluid behavior and also it effects the skin-friction at wall.

Chapter 5:

In the chapter 5, we have considered the boundary layer equations for natural convection laminar incompressible flow of a class of Non-Newtonian fluid past vertical plate. Again the governing system of highly non linear partial differential equations is transformed in to the system of non linear ordinary differential equations using the deductive group theoretic method. For the particular Non-Newtonian fluids namely Powell-Eyring and Williamson, the reduced non linear couple equations are solved by

coding for the suitable solver in MATLAB. The numerical solutions for different flow parameters as well as for different Prandtl numbers are obtained and various profiles are generated. From these solutions, a comparative study on both the fluids, for the effect of flow parameters and influence of Prandtl numbers is discussed.

Chapter 6:

In this chapter we have consider the highly nonlinear system of partial differential equations governing the laminar incompressible flow of Non-Newtonian power-Law fluid past over stretching/shrinking sheet in one attempt. The system of partial differential equation is transformed in the ordinary differential equation by searching the group of transformation subject to the similarity requirement, under the application of one parameter deductive group of transformations. The complete boundary value problem is coded according to the bvp solver in MATLAB and the numerical solutions together with different profiles are obtained for both shrinking and stretching sheet. Further an effect of magnetic field parameter is analyzed on fluid velocity and skin-friction coefficient, while the sheet changing its behavior from shrinking to stretching property.

Chapter 7:

In the final chapter we have probably, first time derived the similarity solutions of unsteady free-convective laminar incompressible boundary layer flow past over vertical plate of all Power law Non-Newtonian fluids. It is interesting to note that the powerful two parameters deductive group theoretic method based on general group of transformation is applied to derive proper similarity transformations for the governing non linear partial differential equation, by reducing three independent variables to one independent variable in one attempt. Further, it is to be observe that numerical solution of the similarity solutions for such a flow of Power Law Non-Newtonian fluids exist only for particular case. The present similarity equation is solved by shooting method, using solver in MATLAB by coding the governing similarity equations for the case of Newtonian and Non-Newtonian fluids. From the numerical solution it can be interpret ate that the influence of Prandtl number play an important role on the flow behavior and heat transfer in natural convection.

4. Aim and Scope of the Thesis:

The main aim of the present thesis is to modify the classical group theoretic technique to modern group theoretic technique so called deductive group theoretic method for the similarity analysis and apply it to find the similarity solutions of various boundary layer flows of Non-Newtonian fluids. Many important extensions are still possible but the purpose of the present work is to demonstrate extension of the conventional group method for Non-Newtonian flow problems.

The equations derived in this thesis can be found in most of the engineering flow problems, particularly two-dimensional steady and unsteady. Thus, present extension with necessary modification of existing techniques adds more application of these techniques which are useful for the treatment of engineering boundary value problems in the research area of the similar kind, which is the scope of the present thesis.

5. List of Publications based on the research work:

1. Darji, R. M., and Timol, M. G. (2011). Similarity solutions of laminar incompressible boundary layer equations of Non-Newtonian viscoinelastic fluids, *Int. J. of Math. Arch. 2(8):* 1395-1404.

2. Darji, R. M., and Timol, M. G. (2011). Deductive Group Theoretic Analysis For MHD Flow Of a Sisko Fluid In A Porous Medium, *Int. J. of Appl. Math and Mech. 7 (19):* 49-58.

3. Darji, R. M., and Timol, M. G. (2011). Group -Theoretic Similarity Analysis for Natural Convection Boundary Layer Flow of a Class of Non-Newtonian Fluids, *Int. J. of Appl. Adv. Sci. and Tech. Research, 1 (2):* 289-311.

4. Darji, R. M., and Timol, M. G. (2011). Similarity analysis for unsteady free convective boundary layer flow of power law fluids via group theory, *J. of Adv. Math. and Appli. , American sci. publi. (communicated)*

5. Darji, R. M., and Timol, M. G. (2012). MHD Stagnation Point Flow of Non-Newtonian Fluid Over a Non Linearly Stretching/Shrinking Sheet, *(communicated)*

6. Darji, R. M., and Timol, M. G. (2012). Group Invariance Analysis of Boundary Layer Equations of a Special Non-Newtonian Fluid Over a Stretching Sheet, *(communicated)*

List of Research papers presented in International conferences

1. Darji, R. M., and Timol, M. G. (2011). MHD Stagnation Point Flow of Non-Newtonian Fluid Over a Non Linearly Stretching/Shrinking Sheet, *56th congress of ISTAM @ SVNIT-SURAT, Dec 19-21.*

2. Darji, R. M., and Timol, M. G. (2011). Group Invariance Analysis of Boundary Layer Equations of a Special Non-Newtonian Fluid Over a Stretching Sheet, *14th International conference CONIAPS XIV @ SVNIT-SURAT Dec 22-24.*

_____ _____

(Rajesh M. Darji.) (Dr. M. G. Timol.)

7 References

Bird, R. B., Stewart, W. E., and Lightfoot, E. M. (1960). *Transport phenomena.* New York: John Wiley.

Birkeland, B. (1996). *Ludwig Sylow's lectures on algebraic equations and substitutions,, 1862.* Christiania (Oslo): Historia Mathematica (23rd edition).

Birkhoff, G. (1950). *Hydrodynamics.* New Jersey: Princeton Press.

Blasius, H. (1908). Boundary layers in fluids with little viscosity. *NACA-TM* , 1256.

Bluman, G. W., and Anco, S. (2002). *Symmetry and Integration Methods For Differential Equations.* New York: Springer Verlag.

Bluman, G. W., and Cole, J. D. (1974). *Similarity Methods for Differential Equations Appl. Math. Sci. No. 13.* New York: Springer Verlag.

Bluman, G. W., and Kumai, S. (1989). *Similarities and differential equations.* New York: Springer Verlag.

Buckingham, E. (1914). Model experiments and the form of empirical equations. *Trans. ASME, Vol (37)* , 263.

Cohen, A. (1911). *An introduction to Lie Theory of one parameter groups (with application to the solution of differential equations)N.Y. (Reprinted 1931).* New york: G. E. Stechert and Co.

Desai, M. I., and Timol, M. G. (1988). Magnetohydrodynamic of viscous non-Newtonian fluids. *Indian J. Eng. Mater. Sci. Vol. (5), No. 6* , 457-464.

Drachner. (1999). *Application of Lie's theory of ordinary and partial differential equations.* UK: I. O. P. Bristol.

Hansen, A. G. (1964). *Similarity analysis in engineering boundary value problems.* New york: Prentice- Hall.

Hansen, A. G., and Herzig, H. Z. (1956). On Possible Similarity Solutions for Three-Dimensional Incompressible Laminar Boundary Layers. I - Similarity with Respect to Stationary Rectangular Coordinates,. *NACA TN 3768* .

Helmholtz, H. (1973). Uber discontinuierliche flussigkeitsbewegungen. *Montasber Berlin, AKADA* , 215.

Herzig, H. Z., and Hansen, A. G. (1956a). On Possible Similarity Solutions for Three-Dimensional Incompressible Laminar Boundary Layers. II - Similarity with Respect to Stationary Polar Coordinates. *NACA TN 3832* .

Lie, S. (1975). Math. Annalen,Vol. (8). *Math. Annalen, Vol. (8)* , 220.

Malek, M. B., and Badran, N. A. (1991). Group method analysis of steady free-convective laminar boundary-layer flow on a non-isothermal vertical circular cylinder. *J. Comput. Appl. Math. Vol. (36)* , 227–238.

Malek, M. B., and Badran, N. A. (1990). Group method analysis of unsteady free-convective laminar boundary-layer flow on a non-isothermal vertical circular cylinder. *Acta Mech. Vol. (85)* , 193–206.

Malek, M. B., and Helal, M. M. (2006). Group method solution for solving nonlinear heat diffusion problems. *Applied Mathematical Modeling Vol. (30)* , 930–940.

Malek, M. B., and Helal, M. M. (2008). Similarity solutions for magneto-forced-unsteady free convective laminar boundary-layer flow. *Journal of Computational and Applied Mathematics vol. (218)* , 202 – 214.

Malek, M. B., Kassem, M. M., and Mekky, M. I. (2004). Similarity solutions for unsteady free-convection flow from a continuous moving vertical surface. *Journal of Computational and Applied Mathematics, Vol. (11), No. 24* , 164–165.

Moran, M. J., and Gaggioli, R. A. (1968). Reduction of the number of variables in system of partial differential equations with auxiliary conditions. *SIAM J. Appl. Math., vol(16),* , 202-215.

Morgan, A. J. A; (1952). The reduction by one of the number of independent variables in some systems of nonlinear partial differential equations. *Quart. J. Math. Oxford, Vol (3)* , 250–259.

Na, T. Y., and Hansen, A. G. (1967a). Similarity solutions of a class of laminar three-dimensional boundary layer equations of power-law fluids. *Int. J. Non-linear Mech., vol. (2)* , 373-385.

Na, T. Y., Abbott, D. E., and Hansen, A. G. (1967). *Similarity Analysis of Partial Differential Equations.* Dearborn: The University of Michigan.

Pakdemirli, M. (1993). Boundary layer flow of power-law fluids past arbitrary profiles. *IMA J. Appl. Math. Vol. (50)* , 133-148.

Pakdemirli, M. (1994). Similarity analysis of boundary layer equations of a class of non-Newtonian fluids. . *Int. J. Non-Linear Mech. Vol. (29)* , 187-196.

Pakdemirli, M. (1992). The boundary layer equations of third grade fluids. *Int. J. Non-Linear Mech. Vol. (27)* , 785-793.

Pakdemirli, M., and Boyaci, H. (2002). Effect of non-ideal boundary conditions on the vibrations of continuous systems. *Journal of Sound and Vibration, Vol. (249), No. 4* ,

815–823.

Pakdemirli, M., and Yurusoy, M. (1999). Equivalence transformations applied to exterior calculus approach for finding symmetries: an example of non-Newtonian fluid flow. *Int. J. of Engineering Science, Vol. (37)* , 25-32.

Pakdemirli, M., and Yurusoy, M. (1997). Symmetry reduction o unsteady three-dimensional boundary layers of some non-Newtonian fluids. *, Int. J. Engng Sci. Vol. (35), No. 8* , 731-740.

Patel, M., and Timol, M. G. (2008). Similarity solutions for three-dimensional boundary layer small-cross flow of Power-law fluids. *International Journal of Mathematical Sciences and Engineering Application (IJMSEA) Vol. (2), No. 1* , 167-174.

Reynolds, O. (1895). On the dynamical theory of incompressible viscous fluids and the determination of the criterion. *Philos. Trans. R. Soc. vol (186)* , 123-164.

Rogers, C., and Ames, w. F. (1989). *non linear boundary value problems in science and engineering.* London: Academic press, inc.

Sedov, L. I. (1959). *Similarity and dimensional Methods in Mechanics.* New york: Academic Press. Inc.

Seshadri, R., and Na, T. Y. (1985). *Group invariance in engineering boundary value problems.* New York: Springer Verlag.

Sirohi, V., Timol, M. G., and Kalthia, N. L. (1987). Powell-Eyring model flow near an accelerated plate. *accelerated plate, Fluid Dy. Res. Vol (2), No.3,* , 193-204.

Timol, M. G., and Kalthia, N. L. (1986). Similarity solutions of three dimensional boundary layer equations of non-Newtonian fluids. *Int. J. Non-Linear Mech. Vol. (21)* , 475-481.

Timol, M. G., and Patel, M. (2009). Numerical treatment of Powell–Eyring fluid flow using Method of Satisfaction of Asymptotic Boundary Conditions (MSABC). *; Applied Numerical Mathematics Vol. (59), Elsevier, North- Holland* , 2584–2592.

Timol, M. G., and Patel, M. (2007). On the class of similarity solutions for three dimensional boundary layer flows of non-Newtonian fluids. *, J. of Veer Narmad South Gujarat University, II-B* , 103-109.

Timol, M. G., and Timol, A. G. (1988). Three dimensional Magnetofluiddynamic flow with pressure gradient and fluid injection. *Ind. J. Pure and Appl. Maths, Vol. (19)* , 208-216.

Wilkinsion, W. L. (1960). *Non-Newtonian fluids.* New York: Pergamon press.